Riemannian manifolds

(M,g) : M - n-dimensional manifold equipped with a tensor field $g \in \mathfrak{X}^0_2(M)$ s.t.

$$\left.\begin{array}{l} 1^\circ \ g \text{ is symmetric } g(X,Y)=g(Y,X) \\ 2^\circ \ g(X,X)=0 \Leftrightarrow X=0 \\ 3^\circ \ g(X,X)\geq 0 \end{array}\right\} \forall X,Y \in \mathfrak{X}(M)$$

is called <u>Riemannian manifold</u>

<u>Rmk 1</u> In physics a weaker structure is usually used for which 2° is replaced by

$$2^{\circ\prime} \ \big(g(X,Y)=0 \ \ \forall Y \in \mathfrak{X}(M)\big) \Rightarrow X=0$$

and 3° is abandoned.

(M,g) with 1°, $2^{\circ\prime}$ is called <u>pseudo-Riemannian manifold</u>

<u>Rmk 2</u> of course 2° implies $2^{\circ\prime}$ since if $2^{\circ\prime}$ was not satisfied we would have $X \neq 0$ s.t. $g(X,Y)=0 \ \ \forall Y \in \mathfrak{X}(M)$. In particular $g(X,X)=0$ with $X\neq 0$, contradicting 2°.

<u>Rmk 3</u> Locally $g, (X_\mu)$: $g(X_\mu, X_\nu) = g_{\mu\nu}$ - functions in U s.t. $g_{\mu\nu}=g_{\nu\mu}$. At every point $p\in U$ by a $GL(n,\mathbb{R})$ transformation can be brought to $g_{\mu\nu} = (\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q})$

Can not change from point to point without violation of continuity.

$\boxed{\text{In Riemannian case } q=0}$ (p,q) is called signature of g at p.

Definition

differentiable ≡ class C^∞

M^n - manifold of dim n.

1) (M,g), (M',g') two (pseudo) Riemannian manifolds

$$\phi: M \xrightarrow{\text{diffeo}} M'$$

is called isometry iff

$$g_p(X_p, Y_p) = g'_{\phi(p)}(\phi_{*p}X_p, \phi_{*p}Y_p) \quad \forall X_p, Y_p \in T_pM \quad \forall p \in M$$

2) $\phi: M \to M'$ is called a local isometry at $p \in M$

if there exists $\mathcal{U} \subset M$ of p s.t.

neighbourhood

$\phi: \mathcal{U} \to \phi(\mathcal{U})$ is an isometry between

$(\mathcal{U}, g)$ and $(\phi(\mathcal{U}), g')$.

3) (M,g) is locally isometric to (M',g') if

for every $p \in M$ there exists $\mathcal{U}$ of p and

a local isometry $\phi: \mathcal{U} \to \phi(\mathcal{U}) \subset M'$.

Examples

1) $M = \mathbb{R}^n$. with cartesian coordinates $(x^1, \dots, x^n)$

$$\Rightarrow \quad g = dx^{1^2} + \dots dx^{n^2}$$

$$dx^{\mu^2} = dx^\mu \otimes dx^\mu \quad \left(\begin{array}{l} \text{notation} \\ dx^\mu dx^\nu = \frac{1}{2}\, dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu \end{array} \right)$$

$(\mathbb{R}^n, g)$ - Euclidean space of dimension n

2) Immersed manifolds

$$\phi : M^n \xrightarrow[\text{immersion}]{} N^{n+k} \quad \left(\begin{array}{l} \phi\text{-differentiable} + \\ \phi_{*p} \text{ injective } \forall p \in M^n \end{array} \right)$$

If N^{n+k} is Riemannian with metric g' we define g by:

$$g_p(X_p, Y_p) = g'_{\phi(p)}(\phi_{*p} X_p, \phi_{*p} Y_p) \qquad \begin{array}{l} \forall p \in M^n \\ \forall X_p, Y_p \in T_p M^n \end{array}$$

Since ϕ is immersion g is positive definite.

In particular if $M \subset (N, g')$ is a submanifold we have

$$\imath : M \xrightarrow[\text{embedding}]{} N$$

$$\Rightarrow \quad g = \imath^* g' \quad \text{is a Riemannian metric on } M$$

$$S^n = h^{-1}(0), \quad h : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$$

$$h(x^\mu) = x^{1^2} + \dots + x^{n+1^2} - 1$$

is a submanifold of $\mathbb{R}^{n+1}$.

Pullback of Euclidean metric g from $\mathbb{R}^{n+1}$ to S^n is a canonical metric on S^n.